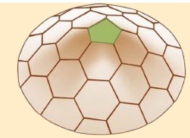
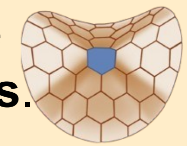
28e Rencontre du Non-Linéaire
25, 26 et 27 mars 2025

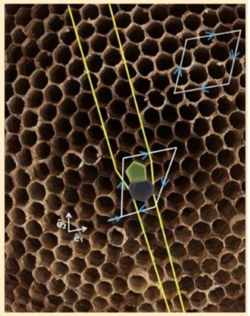
Modeling, Analysis and Finite Element Simulations of Kinematically Incompatible Föppl-von Kármán Plates

Edoardo Fabbrini (fabbrini.edoardo.840@s.kyushu-u.ac.jp)
KYUSHU UNIVERSITY

BACKGROUND & MOTIVATION







Incompatible kinematics: the crystal lattice contains **disclinations**, which are **rotational defects**.

Föppl-von Kármán plates**: nonlinear** plate model that **couples** the **in-plane** (membrane) and **out-of-plane** (transverse) **deformations**.

Motivation: **Tunning** the **mechanical** and **electrical** properties of graphene membranes by **tailoring** the **distribution** of **disclinations**.

THE MECHANICAL MODEL

$\Omega \subset \mathbb{R}^2$ mid-plane of the plate, h plate thickness

$w : \Omega \to \mathbb{R}$ (**out-of-plane displacement**)

$v : \Omega \to \mathbb{R}$ (**Airy stress potential**)

$\nabla^2 v := h \operatorname{cof}(\sigma)$

$\sigma : \Omega \to \mathbb{R}^{2\times 2}_{sym}$ (Cauchy stress tensor)

$\Delta^2 f := \partial_{1111} f + 2\partial_{1122} f + \partial_{2222} f$ (bilaplacian)

$[f, g] := \partial_{11} f \partial_{22} g + \partial_{22} f \partial_{11} g - \partial_{12} f \partial_{12} g$ (Monge-Ampère operator)

$p: \Omega \to \mathbb{R}$ (transverse load)

$\vartheta(x) := \sum_{k=1}^{N} s_k \delta(x - y^{(k)})$ (distribution of **disclinations**)

$$\begin{cases} D\Delta^2 w = [w, v] + p & \text{in } \Omega \\ \dfrac{2}{Eh}\Delta^2 v = -[w, w] + 2\vartheta & \text{in } \Omega \\ w = \partial_n w = 0 & \text{on } \partial\Omega \\ v = \partial_n v = 0 & \text{on } \partial\Omega \end{cases} \quad (1)$$

THE ANALYTICAL RESULTS

Theorem 1 (**Existence**) [1]

Let Ω be open, bounded, simply connected set with **Lipschitz boundary**, $p \in H^{-2}(\Omega)$. Then **(1) admits a solution** in $H_0^2(\Omega) \times H_0^2(\Omega)$

Theorem 2 (**Regularity**) [1]

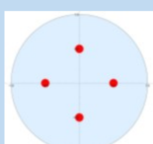
Let Ω be open, bounded, simply connected set with $C^{4,\gamma}$ boundary ($\gamma \in (0,1)$), $p \in L^{k}(\Omega)$. If $(v, w) \in H_0^2(\Omega) \times H_0^2(\Omega)$ solves (1), then

$$w \in W^{4,k}(\Omega)$$

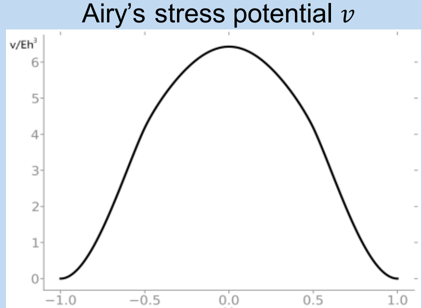
$v \in W^{2,m}(\Omega) \cap C^{4,\gamma}\left(\bar{\Omega} \setminus \bigcup_{k=1}^{N} \overline{B_r(y^{(k)})}\right)$ for $r > 0$ small enough and $m \in [1, \infty)$.

THE NUMERICAL RESULTS

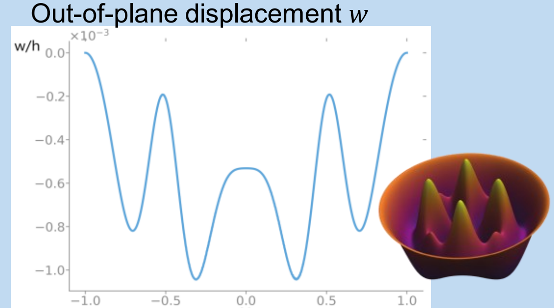
- ✗ **The problem**: conforming FEM for **4th order** PDEs requires $\boldsymbol{C^1}$ FE, **not easy** to implement.
- ✓ **A solution**: use $\boldsymbol{C^0}$ FE, enforce the C^1-regularity by **penalizing** the **discontinuity** of the **gradient** of the shape functions across the facets of the mesh.



Airy's stress potential v



Out-of-plane displacement w



[1] Kinematically incompatible Föppl-von Kármán plates: analysis and numerics. Fabbrini, León Baldelli, Cesana.