On-off intermittency due to parametric Lévy noise

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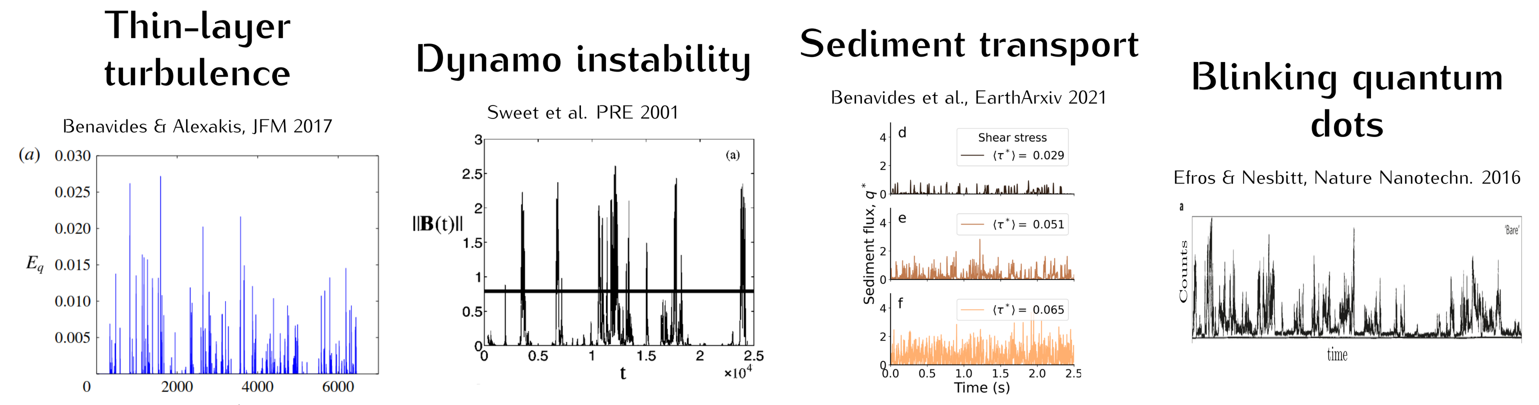
1. Multiplicative noise and on-off intermittency

- Instabilities arise in many systems at a parameter threshold (e.g. onset of convection, 3D instabilities in quasi-2D flows, dynamo instability, sediment transport, etc.)
- Typically, the system is embedded in an uncontrolled noisy environment.
- The fluctuating properties of the environment affect the control parameters of the instability, which leads to parametric (also known as multiplicative) noise.
- Parametric noise close to an instability threshold ⇒ on-off intermittency, switching aperiodically between a large-amplitude "on" state and a small-amplitude "off" state.



- The noisy supercritical pitchfork bifurcation gives a minimal example of this behaviour

$$\dot{X} = (\mu + f(t))X - \gamma X^3, \qquad (1)$$

with mean growth rate μ, nonlinear coefficient $\gamma \geq 0$, and the random noise $f(t)$.

2. (Generalised) central limit theorem & stable laws

- Typically $f(t)$ in (1) is taken to be Gaussian white noise. This is motivated by the **CLT**:

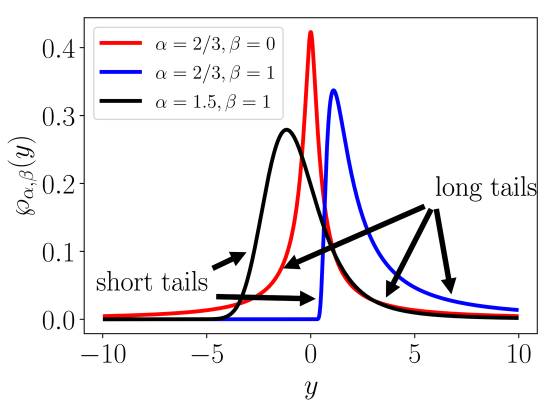
> Given N identically distributed RVs $X_1, \dots, X_N$ with mean 0 and variance σ^2, then $S_N = (X_1 + \dots + X_N)/\sqrt{N} \overset{N\to\infty}{\longrightarrow} S \sim \mathcal{N}(0, \sigma^2)$, if and only if
> 1) The X_i are mutually independent, i.e. $\langle X_i X_j \rangle = 0$ for $i \neq j$ and
> 2) The X_i have finite variance.

- Both assumptions of the CLT may be violated when choosing $f(t)$.
  1) Finite correlation time ⇒ spectrum at zero frequency important [1].
  2) Infinite variance ⇒ noise from non-equilibrium source (no temperature)
- **Generalised CLT** for 2): the scaled sum of the X_i tends to a stable distribution $\wp_{\alpha,\beta}(x)$,

$$\varphi_{\alpha,\beta}(k) = \langle e^{ikx} \rangle = \int_{-\infty}^{\infty} e^{ikx} \wp_{\alpha,\beta}(x) dx = \exp\left\{ -|k|^\alpha \left(1 - i\beta \tan\left(\frac{\alpha\pi}{2}\right)\right)\right\}.$$

- Some simple properties of stable distributions

→ Choosing $\alpha = 2$ gives Gaussian
→ $\wp_{\alpha,\beta}(x) \geq 0 \Rightarrow 0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$.
→ For $|\beta| < 1$, $\wp_{\alpha,\beta}(x) \overset{x\to\pm\infty}{\sim} (1 + \beta \mathrm{sign}(x))x^{-1-\alpha}$.
→ This breaks down on one side for $\beta = \pm 1$.
There, $\wp_{\alpha,\beta}(x) \propto \exp\left(-cst.\ x^{\frac{\alpha}{\alpha-1}}\right)$
⇒ one-sided distribution for $\beta = \pm 1$, $\alpha < 1$.



- Motion driven by stable white noise ⇒ Lévy flight

3. The fractional Fokker-Planck equation

- For (1) with α-stable white noise ($f(t)dt = dt^{1/\alpha}F(t)$, $F(t)$ α-stable), the PDF of X obeys

$$\partial_t p_y(y,t) = -\partial_y[(\mu - \gamma e^{2y})p_y(y,t)] + \mathcal{D}_y^{\alpha,\beta} p_y(y,t), \qquad (2)$$

with $Y = \log(X)$, the (Stratonovich) **fractional Fokker-Planck equation**, and a linear operator $\mathcal{D}_x^{\alpha,\beta}$ (Riesz-Feller fractional derivative). The variable Y performs a **Lévy flight**.
- For $\alpha = 2$ (Gaussian white noise), $\mathcal{D}_x^{\alpha,\beta} = \partial_x^2$. There for $\mu > 0$, the stationary PDF is

$$p_{st}(x) = Nx^{-1+\mu} e^{-\frac{\gamma}{2}x^2} \qquad (3)$$

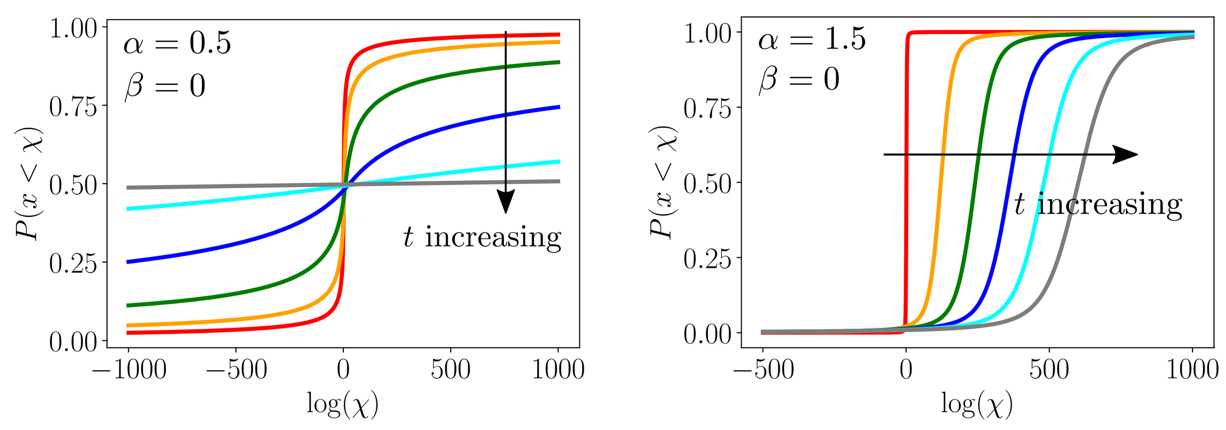
Some important properties
→ Critical transition at $\mu = 0$ (deterministic threshold)
→ Power law divergence at small x with exponent → -1 as $\mu \to 0$, cut-off at large x
→ Anomalous scaling near onset: for all $n > 0$, $\langle X^n \rangle \propto \mu$ as $\mu \to 0$
- **Goal**: extend this result to $\alpha < 2$.
- **Problem**: Can only solve (2) analytically for $\gamma = 0$, the log-stable process,

$$p_x(x,t) = \frac{\wp_{\alpha,\beta}\left(\frac{\log(x) - \mu t}{t^{1/\alpha}}\right)}{x t^{1/\alpha}}, \qquad (4)$$

which does not converge to a stationary state, since probability escapes to $\pm\infty$.
- For $\gamma > 0$, a stationary state exists and its asymptotics can be computed.

4. The linear regime ($\gamma = 0$)

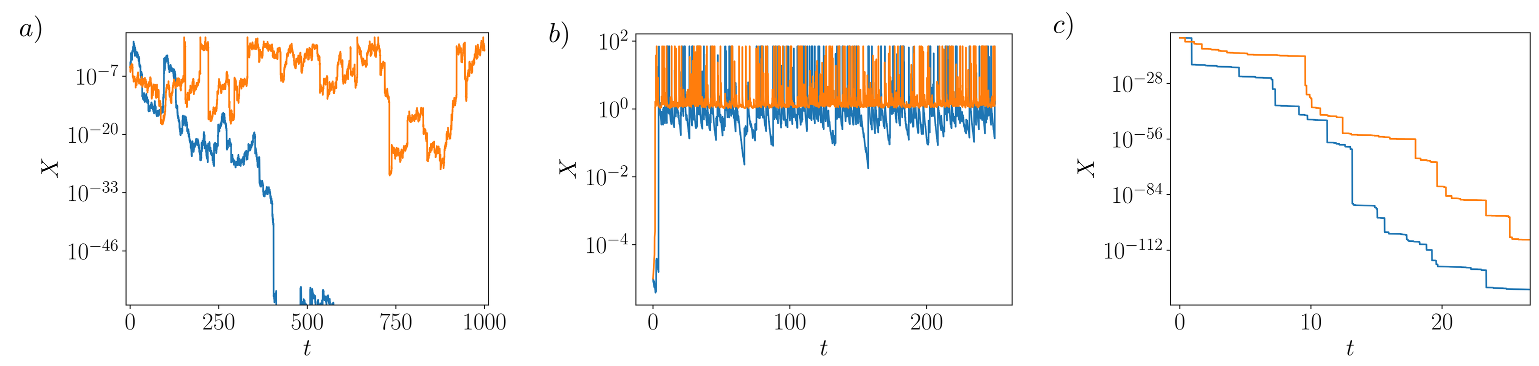
The leakage of probability depends on α, β and μ in this case. E.g. for $\mu > 0, \beta = 0$,



Critical difference: $\alpha > 1$ (mean of noise finite) and $\alpha \leq 1$ (mean of noise infinite)
- For $1 < \alpha < 2$: critical transition at $\mu = 0$ from probability accumulating at $x = 0$ (stable origin) or leaking to $x = \infty$ (unstable origin).
- For $\alpha = 1$, and for $\alpha < 1$, $\beta < 1$, the origin is always stable
- For $\alpha < 1$, $\beta = 1$ (noise positive definite), the origin is always unstable

5. The nonlinear regime ($\gamma > 0$)

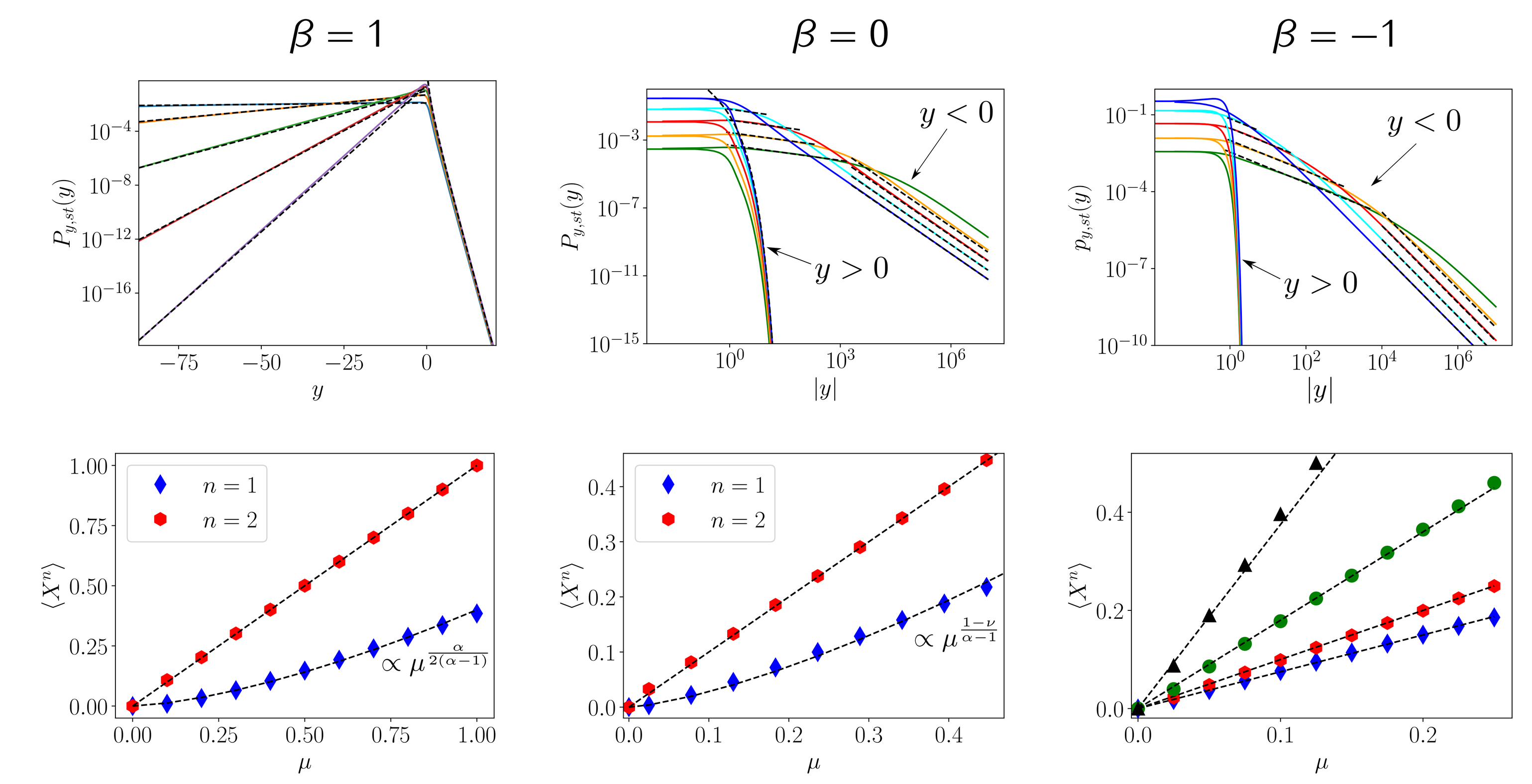
Typical time series a) $\alpha = 1.5, \beta = 0$, b) $\alpha = 0.5$, $\beta = 1.0$, c) $\alpha = 0.5$, $\beta = -1$.



A critical transition only occurs for $\alpha > 1$. Else, the origin is either always stable/unstable.
The exact asymptotics of $p_{x,st}$ in steady state are summarised in the table below

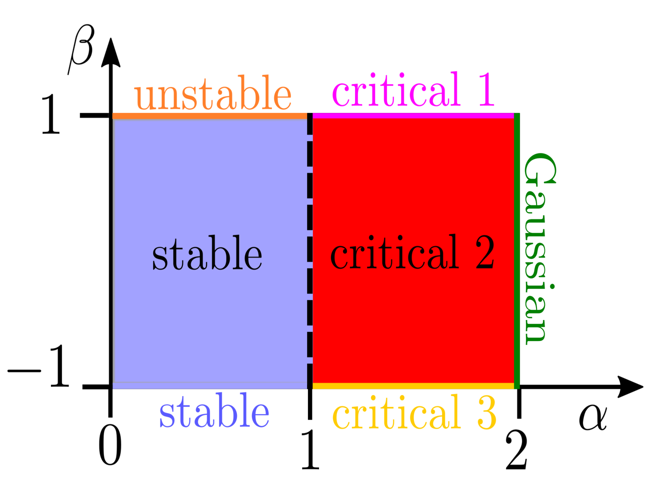
β	$p_{x,st}(x \to 0)$	$p_{x,st}(x \to \infty)$
-1	$C(\mu x)^{-1} \log^{-\alpha}(1/x)$	exponential decay
$(-1, 1)$	$C(\mu x)^{-1} \log^{-\alpha}(1/x)$	$C\gamma^{-1}x^{-3}\log^{-\alpha}(x)$
1	$\propto x^{-1+A_\alpha(\mu)}$	$C\gamma^{-1}x^{-3}\log^{-\alpha}(x)$

Numerically integrating (2) confirms asymptotics ⇒ predict critical exponents (heuristic)



6. Conclusion and Outlook

- Different anomalous critical exponents and stationary PDFs compared to Gaussian noise.
- First step in the study of instabilities in the presence of multiplicative Lévy noise. Many directions can be further pursued, including truncated Lévy noise, combined Lévy-Gaussian noise, finite-velocity Lévy walk, different nonlinearities, higher dimensions and time statistics.
- Since Lévy statistics are found in many physical systems, the anomalous critical exponents predicted here for instabilities subject to Lévy noise may be observable experimentally.



References

[1] Aumaitre et al. *Low-Frequency Noise Controls On-Off Intermittency of Bifurcating Systems*. PRL 2005
[2] A. van Kan et al. *Lévy on-off intermittency*. arXiv:2102.08832, 2021