Committor Functions for Climate Phenomena at the Predictability Margin: the example of an academic model for ENSO

Dario Lucente[1], George Miloshevich[1], Corentin Herbert[1], and Freddy Bouchet[1]

[1]Univ Lyon, ENS de Lyon, Univ Claude Bernard, CNRS, Laboratoire de Physique, F-69342 Lyon, France

Prediction Problem and Committor Function

Committor Function: let $\mathbf{x} \in \Gamma$ be a point in the phase space Γ and $\{\mathbf{X}(t)\}_{0\le t\le\infty}$ a stochastic process taking values in Γ.
The *first hitting time* of a set $C \subset \Gamma$ is

$$\tau_C(\mathbf{x}) = \inf\{t : \mathbf{X}(t) \in C \mid \mathbf{X}(0) = \mathbf{x}\}. \tag{1}$$

The *committor function* $q(\mathbf{x})$ is the probability that the first hitting time of set $B \subset \Gamma$ is smaller than the first hitting time of set $A \subset \Gamma$[4, 7, 1]:

$$q(\mathbf{x}) = \mathbb{P}(\tau_B(\mathbf{x}) < \tau_A(\mathbf{x})). \tag{2}$$

Prediction Problem: we are interested in computing the probability that an observable $O(\mathbf{X}, t)$ of the system reaches a given threshold ϵ within a fixed time T, i.e. $\mathbb{P}(\max_{0\le t\le T} O(\mathbf{X}(t), t) > \epsilon \mid \mathbf{X}(0) = \mathbf{x})$. This is a committor function for an auxiliary process $\mathbf{Y}_t = \{O(\mathbf{X}_t, t), t\}$[3], where the sets A and B are defined as

$$A = \{(z, t)\,;\, z > \epsilon\,;\, t \in [0, T]\},$$
$$B = \{(z, T)\,;\, z < \epsilon\}. \tag{3}$$

Using these definitions for the sets, we obtain

$$\mathbb{P}\left(\max_{0\le t\le T} O(\mathbf{X}(t), t) > \xi \mid \mathbf{X}(0) = \mathbf{x}\right) = \mathbb{P}(\tau_A(\mathbf{x}) < \tau_B(\mathbf{x})) = q(\mathbf{x}). \tag{4}$$

The Jin and Timmermann model

The Jin and Timmermann model is toy model for ENSO [5, 6]. The dimensionless equations are

$$\dot{x} = \rho\delta(x^2 - ax) + x(x + y + c - c\tanh(x + z)) - D_x(x, y, z)\xi_t,$$
$$\dot{y} = -\rho\delta(x^2 + ay) + D_y(x, y, z)\xi_t, \tag{5}$$
$$\dot{z} = \delta(k - z - \frac{x}{2}),$$

where $\xi(t)$ is a Gaussian white noise and

$$D_x(x, y, z) = [(1 + \rho\delta)x^2 + xy + cx(1 - \tanh(x + z))]\sigma,$$
$$D_y(x, y, z) = \rho\delta x^2\sigma,$$
$$[\delta, \rho, c, k, a] = [0.225423, 0.3224, 2.3952, 0.4032, 7.3939].$$

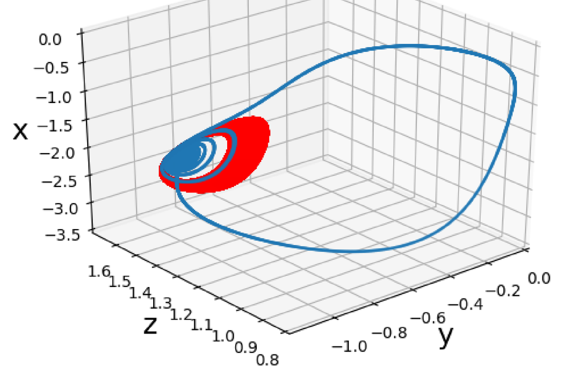

Fig. 0: The two intertwined attractors of the Jin and Timmermann model (limit cycle in blue and strange attractor in red).

Strong El-Niño events if $x > \epsilon = -1$.
$\sigma = 0$ →Two different attractors [2]: one limit cycle that contains strong El-Niño events, one strange attractor without El-Niño events. The two attractors are intertwined with each other (Fig. 0).
$\sigma \ll \sigma_c$ →Switching between two attractors. The occurence of the next El-Niño event is then stochastic.
$\sigma \gg \sigma_c$ →Dynamics is completely dominated by the noise. Distinction between attractors becomes meaningless.

Committor Functions for the Jin and Timmermann model

$\mathbf{x} = (x, y, z)$ is the vector of the model phase space and $\{\mathbf{X}(t)\}_{0\le t\le T}$ is one realisation of the dynamics.
Observable $\mathcal{O}$: x component. **Threshold** $\epsilon = -1$. **Sets** $A = \{(\mathbf{x}, t)\,|\,t \ge T\}$ and $B = \{(\mathbf{x}, t)\,|\,x > \epsilon\}$.

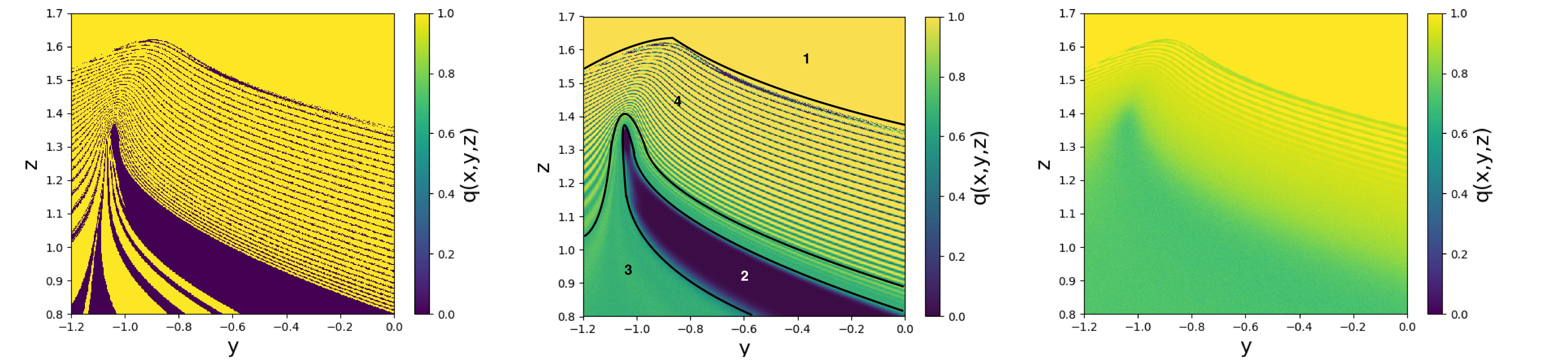

Fig. 1: Colour plot of the committor function $q(x, y, z)$ in the plane $x = -2.8310$, for three values of the noise amplitude, $\sigma = 0$ (left, deterministic), 0.00005 (middle) and 0.001 (right). Regions with uniform $q = 0$ or 1 values correspond to *deterministic predictability*, smooth regions with $0 < q < 1$ to *probabilistic predictability*, and regions with sensitive dependence on initial conditions to unpredictable parts of phase space.

Left panel Fig. 1 $\sigma = 0 \to q(\mathbf{x}) = 0$ or $q(\mathbf{x}) = 1$ (deterministic dynamics). We see 3 regions:

1. large values of z, in a large yellow area all trajectories reach the threshold and $q(\mathbf{x}) = 1$,
2. in a thick black band, no trajectory reaches the threshold and $q(\mathbf{x}) = 0$,
3. everywhere else, we see very fine filaments of alternating yellow and black values. In this area, because of the sensitive dependence on the initial conditions, the occurence of El-Niño is very difficult to predict.

Middle panel Fig. 1 (intermediate value of noise) $\sigma = 0.00005 \to 4$ regions:

1. two regions (1 and 2) of perfect predictability, where the event will occur with probability 1 or 0, respectively,
2. region (3) with good predictability properties where a value $q(\mathbf{x})$ can clearly be predicted with very mild dependence with respect to initial conditions. We call this area the probabilistically predictable region,
3. region (4) which are unpredictable in practice, the strong dependence with respect to the initial condition prevent any practical prediction, either deterministic or probabilistic.

Regions 1), 2) and 4) are reminiscent of their deterministic counterparts. Region 3) is a region where the stochasticity is large enough to smooth out the deterministic values of $q(\mathbf{x})$.
Right panel Fig. 1 (large value of noise) $\sigma = 0.001 \to$ The deterministic predictability is lost for most initial points ($q(\mathbf{x}) \ne 0$ and $q(\mathbf{x}) \ne 1$), the committor function is smooth nearly everywhere, the occurence of El-Niño is probabilistically predictable.

Statistics of the first exit times for transitions to strong El Niño regimes

We define *first exit times* from a point $\mathbf{x}$ to the strong El Niño regime as

$$\tau_c(\mathbf{x}) = \inf\{t > 0 : x(t) > x_c \mid \mathbf{X}(0) = \mathbf{x}\}. \tag{6}$$

The mean first exit time $\mathbb{E}[\tau_c]$ is defined as

$$\mathbb{E}[\tau_c] = \int \mathrm{d}\mathbf{x}\, \rho_{\mathrm{SRB}}(\mathbf{x})\mathbb{E}_{\mathrm{noise}}[\tau_c(\mathbf{x})]. \tag{7}$$

where $\mathbb{E}_{noise}[\cdot]$ is the expectation with respect to the noise realization and $\mathrm{d}\mathbf{x}\,\rho_{\mathrm{SRB}}(\mathbf{x})$ is the SRB measure.
Typically $\tau_c(\mathbf{x}) \gg$ Relaxation time $\to \mathbb{E}[\tau_c] \simeq \mathbb{E}_{SRB}[\tau_c(\mathbf{x})] \simeq \mathbb{E}_{noise}[\tau_c(\mathbf{x})]$ for generic points $\mathbf{x}$ close to the strange attractors.
The probability density function of τ_c is close to an exponential: $p(\tau_c) = \lambda e^{-\lambda\tau_c}$. The parameter λ is equal to the inverse of the mean first exit time: $\lambda^{-1} = \mathbb{E}[\tau_c]$.
In Fig. 2, we show the mean first exit time $\mathbb{E}[\tau_c]$ as a function of the noise amplitude σ.
Fig. 2 clearly shows the mean exit time from the strange attractor to the regime of strong Niño events does not follow an Arrhenius law.
The mean exit time seems much closer to a power law $\mathbb{E}[\tau_c] \propto \sigma^{-1}$.
Two possible heuristic explanations for this interesting breakdown:

- a finite distance $d > 0$ and a quasipotential barrier $\Delta V > 0$ exist, but they are extremely small.
- For any small values d and v, there always exist points in the strange attractor and in the boundary of the basin of attraction at a distance smaller than d and a quasipotential differences ΔV smaller than v.

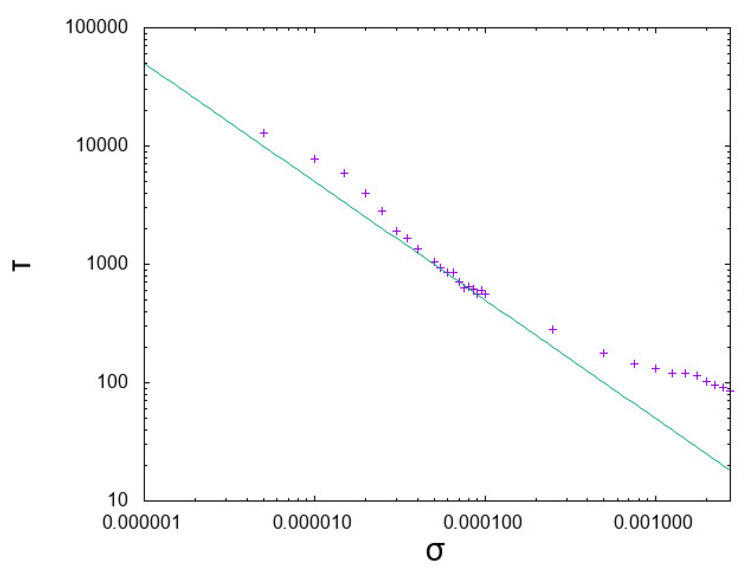

Fig. 2: Mean first exit time $\mathbb{E}[\tau_c]$ for the transition from the strange attractor regime to the strong Niño event regime, as a function of the noise amplitude σ, in log-log coordinates. In the limit of small noise amplitude $\sigma < 10^{-4}$, $\mathbb{E}[\tau_c]$ seems to be closer to a power-law σ^{-1} (green line) than to the standard Arrhenius law.

References

[1] Weinan E, Weiqing Ren, and Eric Vanden-Eijnden. "Transition pathways in complex systems: Reaction coordinates, isocommittor surfaces, and transition tubes". In: *Chemical Physics Letters* 413.1-3 (2005), pp. 242–247. DOI: 10.1016/j.cplett.2005.07.084.

[2] John Guckenheimer et al. "(Un) predictability of strong El Niño events". In: *Dynamics and Statistics of the Climate System* 2.1 (2017), dzx004. DOI: 10.1093/climsys/dzx004.

[3] Thibault Lestang et al. "Computing return times or return periods with rare event algorithms". In: *J. Stat. Mech.* 2018.4 (2018), p. 043213. DOI: 10.1088/1742-5468/aab856.

[4] Dario Lucente et al. "Machine learning of committor functions for predicting high impact climate events". In: *Proceedings of the 9th International Workshop on Climate Informatics: CI 2019* (Paris). Ed. by Julien Brajard et al. NCAR, 2019. DOI: 10.5065/y82j-f154. eprint: 1910.11736.

[5] Axel Timmermann and Fei-Fei Jin. "A nonlinear mechanism for decadal El Niño amplitude changes". In: *Geophys. Res. Lett.* 29.1 (2002), pp. 3–1. DOI: 10.1029/2001GL013369.

[6] Axel Timmermann, Fei-Fei Jin, and Jan Abshagen. "A nonlinear theory for El Niño bursting". In: *J. Atmos. Sci.* 60.1 (2003), pp. 152–165. DOI: 10.1175/1520-0469(2003)060<0152:ANTFEN>2.0.CO;2.

[7] Eric Vanden-Eijnden. "Transition path theory". In: *Computer Simulations in Condensed Matter Systems: From Materials to Chemical Biology Volume 1*. Springer, 2006, pp. 453–493.