Coupled Logistic Maps, growing surfaces and slow systems

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Résumé. Nous étudions le comportement spatio-temporel de la limite continue d'un ensemble d'applications logistiques couplées sur un echaîne uni-dimensionnelle. Nous montrons que l'équation à dérivés partielles résultante est reliée à l'équation stochastique de croissance de Kardar-Parisi-Zhang et à l'équation de Fisher-Kolmogorov-Petrovskii-Piscounov décrivant la propagation des fronts. Une étude du vecteur de Lyapunov du modèle discret confirme que son comportement spatio-temporel est de type KPZ.

Abstract. We discuss the space and time dependence of the continuum limit of an ensemble of coupled logistic maps on a one dimensional lattice. We show that the resulting partial differential equation has elements of the stochastic Kardar-Parisi-Zhang growth equation and of the Fisher-Kolmogorov-Petrovskii-Piscounov equation describing front propagation. A study of the Lyapunov vector of the discrete model confirms that its space-time behavior is of KPZ type.

1 Introduction

Coupled map lattices are dynamical systems with very different collective spatio-temporal regimes selected by tuning a few parameters that are, typically, local ones controlling the chaoticity of the independent units, and the coupling between different units [1]. There have been several attempts to use statistical mechanics notions to describe their spatiotemporal behavior [2]. Continuous limits have also been considered [3]. In this contribution we summarize the results of our recent investigation of the continuous limit of a one dimensional ring of diffusively coupled logistic maps [7]. We discuss its connection with (i) the stochastic Kardar-Parisi-Zhang (KPZ) growth partial-differential equation [4]; (ii) the deterministic Fisher-Kolmogorov-Petrovskii-Piscounov (FKPP) partial differential equation for the spreading of a favorable mutation in the form of a wave [5,6].

2 The continuum limit of the coupled map lattice

The logistic map is a non-linear evolution equation acting on a continuous variable x taking values in the unit interval $[0, 1]$:

$$x_n = f(x_{n-1}) \equiv r\, x_{n-1}(1 - x_{n-1}) \; . \tag{1}$$

The discrete time is labeled by $n = 0, 1, \ldots$. The parameter r takes values in $[0, 4]$. The time series has very different behavior depending on the value of r. For $0 \leq r < 1$ the iteration approaches the fixed point $x^* = 0$. For $1 \leq r < 3$ the asymptotic solution takes the finite value $x^*(r) = 1 - 1/r$ for almost any initial condition. Beyond $r = 3$ the asymptotic solution bifurcates, x_n oscillates between two values x_1^* and x_2^*, and the solution has period 2. Increasing r other bifurcations appear at sharp values. Very complex dynamic behavior arises in the range $r \in [3.57, 1]$: the map has bands of chaotic behavior, *i.e.* different initial conditions exponentially diverge, intertwined with windows of periodic behavior.

A coupled logistic map lattice is a discrete array of coupled continuous variables, x_n^i, each of them evolving in time following (1). A typical interaction that we use here is a nearest neighbors spatial coupling of Laplacian form :

$$x_n^i = f(x_{n-1}^i) + \frac{\nu}{2}\left[f(x_{n-1}^{i-1}) - 2f(x_{n-1}^i) + f(x_{n-1}^{i+1}) \right] , \tag{2}$$

with $x_n^{i+N} = x_n^i$ for all n with N the number of elements on the ring. The initial condition is usually chosen to be random and thus taken from the uniform distribution on the interval $[0, 1]$ independently on each site. ν is the coupling strength between the nodes and plays the role of a viscosity. In a nutshell, the dynamics of this system is characterized by a competition between the diffusion term, that tends to produce an homogeneous behavior in space, and the chaotic motion of each unit, that favors spatial inhomogeneous behavior due to the high sensitivity to the initial conditions.

The main idea is to take the continuum limit of the CML using the usual discretization of time and space derivatives, *e.g.* $\frac{\partial h}{\partial t} \leftrightarrow \frac{h_{n+1}^i - h_n^i}{\delta t}$, *etc.* with δt the time-step and δx the lattice spacing that equal one in our system of units. The CML of logistic elements then becomes

$$\frac{\partial h}{\partial t} = \frac{\nu r}{2}(1-2h)\,\frac{\partial^2 h}{\partial x^2} - \nu r \left(\frac{\partial h}{\partial x}\right)^2 + (r-1)h - rh^2 , \tag{3}$$

where we called x the coordinate ($i\delta x \to x$), t the time ($n\delta t \to t$), and h the field $[x_n^i \to h(x,t) = h]$.

One immediately notices that eq. (3) looks like a KPZ or FKPP equation but :

(i) By definition the field h is bounded and takes values in the unit interval. Thus, the resulting equation should have an effective confining potential that limits the field to a finite range. The field is not bounded in KPZ but it is in the FKPP equation.

(ii) The elastic term is here multiplied by a field-dependent viscosity

$$\nu(h) \equiv \frac{\nu r}{2}(1-2h) . \tag{4}$$

First, $\nu(h)$ is negative for $h < 1/2$ which implies an instability in the hydrodynamic limit. It was shown in [8] and [9] that in the Kuramoto-Sivashinsky equation a similar instability taps the system and so creates an effective 'noise' leading to a mapping onto the KPZ equation. The confining potential restraints the instabilities caused by $\nu(h) < 0$. Second, if h remains bounded the viscosity takes values on a finite interval. However, we expect the field-dependent bare viscosity to be renormalized at large scales by the effect of the non-linear terms (see below) and thus its precise value seems not to be very important.

(iii) The second, non-linear term is of the form of the one in the KPZ equation with a negative coupling $\lambda \equiv -\nu r$, though the sign of λ should not be important. This term does not exist in the FKPP equation.

(iv) The last two terms read

$$\eta(x,t) \equiv (r-1)h(x,t) - rh^2(x,t) . \tag{5}$$

We notice that these terms are not present in the KPZ equation. In order to compare to the latter we argued that they have a double identity : on the one hand η behaves roughly as a short-range correlated noise in space and time ; on the other hand it can be interpreted as a force derived from a confining potential

$$\eta = -\frac{\partial V(h)}{\partial h} , \qquad V(h) = -\frac{(r-1)}{2}h^2 + \frac{r}{3}h^3 . \tag{6}$$

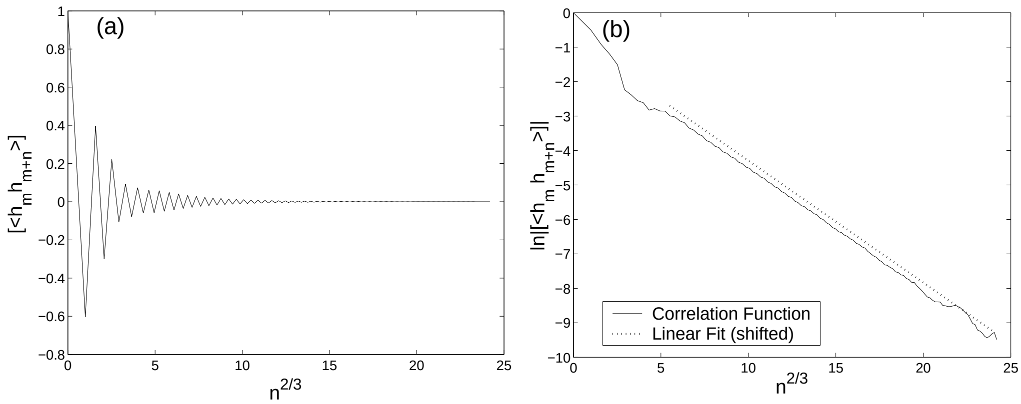
On the other hand, these terms are identical to the ones in the FKPP equation with a particular choice of the parameters in the source terms.

3 Results

Having the connection with KPZ and FKPP equations in mind we integrated numerically the CML of logistic units with up to $N = 1024$ sites and periodic boundary conditions, and we analyzed the space and time behavior of several observables. All units were updated in parallel. We focused on parameters that set the system in the asymptotic chaotic regime (typically, $\nu = 0.4$ and $r = 4$).

First, we analyzed the statistical properties of the 'noise' η_n^i and we found that, even though it is not a perfect random noise it is rather short-ranged correlated in space and time. These terms also provide a confining potential to the surface fluctuations and its roughness is suppressed.

The decay in time of the field-field correlations is still non-trivial and very similar to the one found for the (usual and unbounded) KPZ equation [see, [10,11,12]]. In Fig. 1 we show the space averaged, time correlation of the field $[\langle h_{m+n} h_m \rangle]$.



(a) Linear plot (b) linear-logarithmic scale.

Fig.1. The correlation function $[\langle h_{m+n} h_m \rangle]$ as a function of $n^{2/3}$.

The linear dependence in the linear-logarithmic scale used in panel (b) indicates the stretched exponential decay

$$C_n^h(t) \approx e^{-n^\zeta} , \qquad \text{with} \qquad \zeta = 2/3 . \tag{7}$$

Panel (a) shows that the decay occurs in an oscillatory way. The dependence of the exponent ζ with r is very weak. A number of authors have signalled the possible relevance of CMLs in describing different aspects of glassy relaxation. Recently, [13] studied the same model in its intermittent regime ($\nu = 0.4$, $r \approx 3.83$) with the aim of relating the 10 orders of magnitude stretched exponential decay of its distribution of trapping times (times in which an element remains locked into one of the coarse-grained values $s_n^i = \pm 1$) to the one observed in super-cooled liquids, and they associated this stretched exponential decay to the one of the correlation function. As we have already stressed, we also obtain a stretched exponential relaxation for larger value of r where trapping intervals for the coarse-grained variables s_n^i do not exist.

In the same spirit, we derive a continuum partial differential equation governing the evolution of the Lyapunov vector of that system. The notion of a Lyapunov vector is one of the ways to extend the notion of Lyapunov exponent to space-time chaos. By deriving the continuum equation, we confirm a conjecture by [14,15] that the space-time behavior of the Lyapunov vector becomes the one of the KPZ through a mapping to the Directed Polymer problem. The largest Lyapunov exponent is then obtained from the norm of the Lyapunov vector. f one uses the so-called 0-norm,

$$N_0(t) = \exp\left[\frac{1}{L}\int_0^L h(x,t)dx\right] , \tag{8}$$

it is then assured to be a self-averaging quantity. In addition, the Lyapunov exponent is given by

$$\lambda = \lim_{T\to\infty} \frac{\ln N_0(T) - \ln N_0(0)}{T} , \tag{9}$$

and this is no other than the large-deviation function for the Asymmetric Exclusion Process (ASEP). calculated previously by Derrida and Appert [16]. The ASEP is a discrete model in the universality class as KPZ in one dimension. In terms of the ASEP, the Lyapunov exponent is given by

$$\lambda = \rho(1-\rho) + \sqrt{\frac{\rho(1-\rho)}{2\pi L^3}} G\left(\sqrt{2\pi\rho(1-\rho)L}\right) \tag{10}$$

where L is the system size, ρ is a the density of particles (a parameter in ASEP), and $G(\beta)$ is a scaling function independent of L and ρ and known is an implicit form (see [16] for more details).

This result is useful for studying finite size effects and, more importantly, it can be used to estimate the largest Lyapunov exponent. Doing this we get $\lambda = 9/32 \simeq 0.28$ while in numerical simulations it was found to be $\lambda \simeq 0.38$ [15] which is of the same order of magnitude as our result.

Finally, we analyzed the interpretation of the continuum limit of the CML as a FKPP non-linear diffusion equation with an additional KPZ non-linearity and the possibility of developing traveling wave configurations. We found that the CML do indeed have traveling wave solutions with a velocity that depends on the initial conditions. A more careful analysis is needed to classify them all.

4 Conclusions

Emergence of slow relaxations for this system signals the possible relevance of CMLs in describing different aspects of glassy relaxation, as was already noticed by several authors. With the baggage gained from the current understanding of the dynamics of glassy systems we intend to address generations of effective temperatures in this nonlinear chaotic system and the possible appearance of a fluctuation relation.

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