Nonlinear equation of motion for propagating crack fronts in heterogenousmaterials

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Résumé. Nous avons étudié la propagation d'un front de fissure dans un milieu hétérogène. Dans ce but, nous avons développé au second ordre le facteur d'intensité des contraintes d'un front de fissure courbe, autour de sa solution pour un front droit, en généralisant un premier résultat classique dû à Rice. Nous avons ainsi proposé une équation de mouvement pour la propagation d'un front de fissure dans un milieu hétérogène qui contient deux ingrédients principaux - l'irréversibilité de la propagation du front et les effets non linéaires. En utilisant l'équation stochastique proposée pour le mouvement du front de fissure, nous avons étudié la dynamique de propagation d'un front de fissure dans un milieu hétérogène dans le régime quasi-statique. L'approche consiste à utiliser une expansion auto-consistante (self consistent expansion) introduite par Schwartz et Edwards. Nous avons découvert une transition de phase dynamique continue entre une phase lisse (à grandes échelles) et une phase rugueuse, avec un exposant de rugosité $\alpha = 1/2$.

Abstract. In this work we study the propagation of planar crack fronts in heterogenous materials. For that purpose, we first derive the second order variation in the local static stress intensity factor of a tensile crack with a curved front, thus generalizing the classical first order result of Rice. Using this, we propose an equation of motion for the propagating crack front that contains two main new ingredients - irreversibility of the propagation of the crack front and nonlinear effects. The proposed equation allows for a systematic study of the roughening of the a moving front in the quasi-static regime by using the Self Consistent Expansion, which reveals a rough phase described by a roughness exponent $\alpha = 1/2$.

1 Introduction

The propagation of a crack front in a brittle material is the playground of a number of physical phenomena which range from dynamic instabilities of fast moving cracks [1] to quasi-static instabilities of crack paths [2,3], or of crack fronts [4,5,6,7,8,9]. Although the actual theory of brittle fracture mechanics succeeded to explain a number of instabilities, the experimentally observed self-affine roughness of a crack front propagating through a heterogeneous medium remains the subject of theoretical debate [6,7,8]. This phenomenon is of fundamental importance, because it may be regarded as an archetype of self-affine patterns induced by advancing fronts. Wetting of a disordered substrate being another example of systems with a similar structure [10,11].

In the framework of linear elastic fracture mechanics, an important step was performed by Rice [12] following a work of Meade and Keer [13]. He gave a general formula for the first order variation in elastic fields of a planar curved crack front and subsequent analysis was mainly based on this work [6,7,14,15,16,17]. However, aspects related to crack front roughness and stability could not be derived within this first order perturbation solution. A possible explanation, which has been suggested in the context of the wetting problem [11], is that higher order variations might be necessary for the study of stability and roughening properties of these fronts. For that reason, as a first step in the direction of providing a theoretical explanation we derived a formula for the second order variation in elastic fields of a planar curved crack front [18]. It uses a methodology introduced in [9] for the study of the peeling-induced crack-front instability in a confined elastic film.

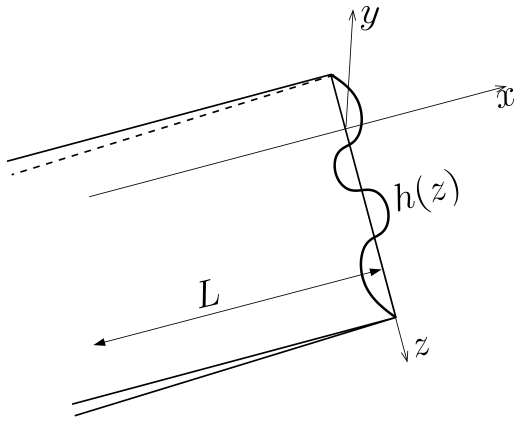

Fig.1. Schematic of the problem of a half-plane crack on $y = 0$ in an infinite body. The average penetration of the crack front in the x-direction is L. The straight reference front in the z-direction and the perturbation $h(z)$ around it are also shown.

2 The second variation in the elastic fields

The problem of a half-plane crack located in the plane $y = 0$ with a curved front (see Fig. 1) can be solved by using the linear equations of elasticity. It has been shown [13] that these equations are satisfied for a tensile loading that is symmetric to the crack plane and so no other components of the stress intensity factors (SIFs), other than K_I are present for that case. From a mathematical point of view, the starting point is one of finding a function Φ satisfying

$$\triangle\Phi(x, y, z) = 0 , \tag{1}$$

having vanishing derivatives at infinity. Defining $h(z)$ to be the position of the crack front, the main challenge is to solve for Φ in the presence of this curved front. This is done essentially by transforming into a set of coordinates defined at the crack front, i.e. from (x, y, z) to $(X \equiv x - h(z), y, z)$ [9]. The result of this calculation is given schematically by

$$K_I(z) = K_{I0} + K_{I1}(z) + K_{I2}(z) + O\left(h^3, \frac{h}{L}\right) , \tag{2}$$

where K_{I0} is just the SIF of a straight crack under the same loading conditions, $K_{I1}(z)$ is the first order variation, derived previously by [12], and $K_{I2}(z)$ is the new result. These terms are given by

$$K_{I1}(z) = PV \int_{-\infty}^{\infty} K_0(z') \frac{h(z') - h(z)}{(z' - z)^2} \frac{dz'}{2\pi} , \tag{3}$$

$$K_{I2}(z) = -\frac{1}{8} K_0(z) h'^2(z) + PV \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_0(z') \frac{(h(z') - h(z))(h(z'') - h(z'))}{(z' - z)^2 (z'' - z')^2} \frac{dz''}{2\pi} \frac{dz'}{2\pi} , \tag{4}$$

Let us emphasize that for the study of the crack front stability, this perturbation expansion is incomplete, because contributions of order (h/L) have been omitted (with L being the average length of the crack in the z direction — see Fig. 1). This statement is true even for a linear stability analysis. An example of the importance of such contributions is given by the linear stability analysis of the peeling-induced crack-front in a confined elastic film [9], where the (h/L) terms do rule the stability of the crack front. From a conceptual point of view these terms are important to keep contact with experiments [5], because a quasistatic moving crack front will always stop $(dK/dL < 0)$, unless the applied force is increased. Indeed the experimental realizations for the study of crack front roughness use the large length scale L in order to make the interface moving, by applying an increasing opening in a cantilever beam configuration. We believe that such effects are also present in wetting experiments, where the contact line is displaced by pulling off the substrate. In such conditions, the roughening of the interface results from a competition between the microscopic pinning effects and the destabilizing effects of the macroscopic driving.

3 The equation of motion

The next step is to propose an equation for the motion of a planar crack in a heterogeneous material. The present approach is very similar to the one introduced by Gao and Rice [14,15,16]. We write the equation of motion for the moving crack front as a stochastic partial differential equation by using two main ingredients - the irreversibility of crack front propagation and the nonlinear effects. Here, we refer to $h(z)$ as the fluctuating part of the interface, so that by definition the real location of the interface is given by $L + h(z)$, and L is its average. The proposed equation contains two important ingredients - irreversibility of the propagation of the crack front and nonlinear effects. It is given by

$$\frac{\partial h}{\partial t}(z,t) = \sqrt{1+h'^2}\left[K_I(h) - K_c(z,h)\right]\Theta\left[K_I - K_c\right] \ , \tag{5}$$

where $h' = \partial h/\partial z$, $\Theta(\cdot)$ is the Heaviside function, $K_I(h)$ is the stress intensity factor of the crack front (given to second order above) and $K_c(z,h)$ is a random term representing the heterogeneity in the local material toughness due to disorder. The random term can always be separated as $K_c(z,h) = K^* + \eta(z,h)$, where K^* is an average toughness and η is its fluctuating part.

Solutions of stochastic growth models such as Eq. (5) exhibit scaling behavior which is described using the time dependent height-height correlation function

$$\left\langle \left[h(z,t) - h(z',t')\right]^2 \right\rangle^{1/2} = |z-z'|^{2\alpha} f\left(\frac{|z-z'|}{|t-t'|^{\mu}}\right), \tag{6}$$

where α (sometimes denoted as ζ) is the roughness exponent of the interface and μ is the dynamic exponent (sometimes denoted as z). The brackets $\langle\cdots\rangle$ denote average over disorder.

For the proposed model we identify three different regimes : A static regime for which $K_0 \ll K^*$ (where the Heaviside function in (5) can be safely approximated by 0) ; A regularly moving interface for large values of K_0 (where the Heaviside function can be safely approximated by 1) ; And an intermediate complex regime, where $K_0 \sim K^*$. In this last regime, a very important factor seems to be the stabilizing terms (i.e. those terms of order h/L that were dropped out in the derivation) that will make sure that the crack will stop after a while (as indeed seen in experiments).

Based on that picture, we hypothesize that a frozen dynamically rough interface is seen in experiments [5]), rather than a rough phase determined by a static pinned interface. In other words, we stress the point that the crack tends to stop due to its physical nature even without the presence of heterogeneities. This is indeed the case in cantilever beam experiments [5], where the crack faces are increasingly opened in order to induce crack front motion. As a result the front starts moving until it stops. The heterogeneities only induce roughness and as we argue, a dynamical roughness, which is then frozen due to the irreversibility of the fracture process.

In order to test this picture, we approximate this system by neglecting consistently all mechanisms which deal with the slowing down of the interface, as well as the freezing of it. The assumption here is that the specific aspect of fine-tuning the opening stress mode (for example by imposing a time-dependent external loading) is exactly what the experiment does. Then we analyze the system at that critical point whichever means were taken to get there. This involves neglecting the Heaviside function on the right-hand side of Eq. (5). We suspect that this term does play a role in the final stages of freezing, namely by imposing differential arrest along the interface (note again that the interface would stop anyway, even without this term). This would tend to increase the roughness. Thus, we would consider the results obtained below as a lower bound for the roughness, offering a quantitative physical explanation up to the last steps of the freezing.

Following the previous arguments, we approximate the noise term for the moving front, where $h \simeq vt$, by $\eta(z,h) \simeq \eta(z,vt) = \hat{\eta}(z,t)$, i.e. as a "thermal noise" [11]. Also, we do keep nonlinear terms, since we claim (and will justify later) that they play an important role in roughening the interface. Obviously a linear equation of the kind described above (i.e. taking into consideration only the linear term in $K_I(h)$) would not yield any roughness, and actually even if the KPZ nonlinearities (i.e. h'^2 terms) are kept, we would also end up with a smooth surface, or at most logarithmically rough (this is a special case of the so called Fractal KPZ equation studied previously in [19]). When keeping consistently second order terms, the resulting equation of motion becomes

$$\frac{\partial h}{\partial t} = K_0 \int_{-\infty}^{\infty} \frac{h'(z')}{(z'-z)} \frac{dz'}{2\pi} + K_0 \int_{-\infty}^{\infty} \frac{dz'}{2\pi} \int_{-\infty}^{\infty} \frac{dz''}{2\pi} \frac{h'(z')h'(z'')}{(z'-z)(z''-z')} - \frac{3}{8}\left(\frac{4}{3}K^* - K_0\right) h'^2 + (K_0 - K^*) + \hat{\eta}(z,t) \ , \tag{7}$$

with noise correlations described by

$$\langle \hat{\eta}(z,t)\,\hat{\eta}(z',t') \rangle = 2D\delta(z-z')\,\delta(t-t') \ , \tag{8}$$

where D is the variance of the noise. The constant term, $(K_0 - K^*)$, in Eq. (7) can be put aside by transforming into a co-moving coordinate system. Moreover, by looking at the KPZ term (i.e. h'^2) we can estimate the region where this discussion is relevant. Roughly, when the coefficient of that term remains negative (i.e. for $K_0 < \frac{4}{3}K^*$), we are still in the quasi-static regime since in that case a rough interface would decrease the velocity, while for higher values of the applied stress ($K_0 > \frac{4}{3}K^*$) the system would be in the regularly moving regime. This estimate is consistent with our assumption that the dynamics of interest is not necessarily at $K_0 \simeq K^*$, but in some range above it.

4 The Self Consistent Expansion

We now apply the self-consistent-expansion (SCE) method to this simplified equation of motion in order to derive results for the scaling exponents [20]. This method was developed by Schwartz and Edwards [21,22] and has been applied successfully to the Kardar Parisi-Zhang (KPZ) equation [23]. The method gained much credit by being able to give sensible predictions for the KPZ scaling exponents in the strong-coupling phase above one dimension where many renormalization group (RG) approaches failed. Another point which is especially relevant for our purpose is that for a family of models with long-range interactions (of the kind treated presently) SCE reproduced exact one-dimensional results while RG failed to do so [24].

The SCE method is based on going over from the Fourier transform of the equation in Langevin form to a Fokker-Planck form and on constructing a self-consistent expansion of the distribution of the field concerned. We then consider the simplified version of the equation of motion in Fourier components

$$\frac{\partial h_q(t)}{\partial t} = -c_q h_q - \sum_{\ell,m} M_{q\ell m} h_\ell h_m + \hat{\eta}_q(t) \, , \tag{9}$$

where $c_q = \frac{K_0}{2}|q|$ and $M_{q\ell m} = -\frac{K_0}{4\sqrt{L_z}}|q|\,|\ell|\,\delta_{q,\ell+m}$, L_z being the linear size of the front. Note that in contrast to the KPZ problem $M_{q\ell m}$ has the symmetries

$$M_{q\ell m} = M_{-q,\ell,m} = M_{q,-\ell,m} = M_{q,\ell,-m}. \tag{10}$$

Last, $\hat{\eta}_q(t)$ is a noise term with zero average described by its variance

$$\langle \hat{\eta}_q(t)\,\hat{\eta}_{q'}(t') \rangle = 2D\delta_{q,-q'}\delta(t-t') \, . \tag{11}$$

Rewriting this equation in a Fokker-Planck form we get

$$\frac{\partial P}{\partial t} + \sum_q \frac{\partial}{\partial h_q}\left[D_0 \frac{\partial}{\partial h_{-q}} + c_q h_q + \sum_{\ell,m} M_{q\ell m} h_\ell h_m\right] P = 0 \ , \tag{12}$$

where $P\left(\{h_q\},t\right)$ is the probability functional for having a height configuration $\{h_q\}$ at time t.

The expansion is formulated in terms of the steady-state structure factor $\phi_q = \langle h_{-q}h_q\rangle$ (or two-point function), and its corresponding steady-state decay rate that describes the rate of decay of a disturbance of wave vector q in steady state, namely

$$\omega_q^{-1} = \frac{\int_0^\infty \langle h_{-q}(0)h_q(t)\rangle\, dt}{\langle h_{-q}h_q\rangle}. \tag{13}$$

From the scaling form (6) it follows that for small q's ϕ_q and ω_q behave as power laws in q, namely $\phi_q = A|q|^{-\Gamma}$ and $\omega_q = B|q|^{\mu}$, where z is the dynamic exponent, and the exponent Γ is related to the roughness exponent by $\alpha = (\Gamma - 1)/2$.

The main idea of SCE is to write the Fokker-Planck equation $\partial P/\partial t = OP$ in the form $\partial P/\partial t = [O_0 + O_1 + O_2]\,P$, where O_0, O_1 and O_2 are zero, first and second order operators in some parameter. The evolution operator O_0 is chosen to have a simple form $O_0 = -\sum\limits_q \frac{\partial}{\partial h_q}\left(D_q \frac{\partial}{\partial h_{-q}} + \omega_q h_q\right)$, where $D_q/\omega_q = \phi_q$. Note that ϕ_q and ω_q are still unknown. Next, an equation for the two-point function is obtained. The expansion has the form $\phi_q = \phi_q + e_q\left\{\phi_p,\omega_p\right\}$, where e_q is a functional of all ϕ's and ω's. This reflects the fact that the lowest order in the expansion is exactly the unknown ϕ_q. In the same way, an expansion for ω_q is given by $\omega_q = \omega_q + d_q\left\{\phi_p,\omega_p\right\}$. Now, the two-point function and the characteristic frequency are determined by setting $e_q\left\{\phi_p,\omega_p\right\} = 0$ and $d_q\left\{\phi_p,\omega_p\right\} = 0$. To second order in the expansion, we get the following two coupled integral equations

$$D_0 - \frac{K_0}{2}\,|q|\,\phi_q + I_1(q)\,\phi_q + I_2(q) = 0\;, \tag{14}$$

$$\omega_q - \frac{K_0}{2}\,|q| + J(q) = 0\;, \tag{15}$$

with

$$I_1(q) = \frac{K_0^2}{32\pi}\,|q| \int d\ell\,|\ell|\,\frac{|\ell|(|q-\ell|+|q|)\phi_{q-\ell}+|q-\ell|(|\ell|+|q|)\phi_\ell}{\omega_q+\omega_\ell+\omega_{q-\ell}}\;, \tag{16}$$

$$I_2(q) = \frac{K_0^2}{32\pi} q^2 \int d\ell\,|\ell|\,\frac{(|\ell|+|q-\ell|)\phi_\ell\phi_{q-\ell}}{\omega_q+\omega_\ell+\omega_{q-\ell}}\;, \tag{17}$$

$$J(q) = \frac{K_0^2}{32\pi}\,|q| \int d\ell\,|\ell|\,\frac{|\ell|(|q-\ell|+|q|)\phi_{q-\ell}+|q-\ell|(|\ell|+|q|)\phi_\ell}{\omega_\ell+\omega_{q-\ell}}\;. \tag{18}$$

It is interesting to mention here that Eq. (14) can be understood as emanating from the short time balance of the original equation, while Eq. (15) comes from its long time balance [22].

These equations can be solved exactly in the asymptotic limit (i.e. for small q's) to yield the required scaling exponents governing the steady-state behavior and the time evolution. We will not get more into technical details, but rather summarize our results. We found two possible phases : First, a flat phase described by $\alpha = 0$ and $\mu = 1$, corresponding to the system in the moving regime. This phase is always possible. Second, we see the possibility of having a rough phase with $\alpha = 1/2$ and $\mu = 2$, which is possible only for some critical values of the forcing at slow velocities.

5 Summary and Discussion

Having in mind the roughness of a propagating crack front in heterogeneous materials we derived the second order variation in the stress intensity factor of a tensile crack with a curved front propagating in a brittle material. We pointed out that for linear stability analysis one has to take into account the contributions coming from the large scales and so the complete resolution of a given problem must be fully performed for that purpose. Then, we proposed an equation of motion of planar crack fronts in heterogeneous media that contains both the irreversibility of the propagation of the crack front and nonlinear effects. We show that the proposed equation can be useful in studying the roughening of propagating crack fronts. We do so by using the method of the self-consistent expansion. We found the possibility of having a rough moving phase with $\alpha = 1/2$ (and $\mu = 2$) which is relevant for $K_0 \sim K^*$ due to destabilization of the nonlocal elasticity by the nonlinear term. This result is in agreement with the roughness exponent measured in experimental systems [4,5]. Since in our analysis we neglected the irreversibility of the fracture process (which becomes important during the last steps of freezing, and so tends to further roughen the line), our analysis provides a lower bound for the experimental results (recall that experimental results vary between 0.5–0.6). We hope that analysis of the full equation would yield results which are even closer to the experimental measurements. On the other hand, an interesting challenge to experiments would be determination of the dynamic exponent μ from direct measurements.

Acknowledgements – This work was supported by EEC PatForm Marie Curie action (E.K.). We thank D. Vandembroucq and A. Boudaoud for fruitful discussions. Laboratoire de Physique Statistique de l'Ecole Normale Supérieure is associated with the CNRS (UMR 8550) and Universities Paris VI and Paris VII.

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